J. Fluid Mech. (2002), *vol.* 455, *pp.* 283–287.
DOI: 10.1017/S0022112001007212 Printed in the United Kingdom

On slow oscillations in coupled wells

By JOHN MILES

Cecil H. and Ida M. Green Institute of Geophysics and Planetary Physics,
University of California, San Diego, La Jolla, CA 92093-0225, USA

(Received 27 July 2001 and in revised form 30 September 2001)

The eigenvalue problem for slow oscillations of a liquid in a set of N cylindrical wells that are bounded above by free surfaces and below by a common, semi-infinite reservoir is formulated on the assumption that the depth of the wells is large compared with their width, so that the lowest mode in each well, for which the fluid moves as a rigid body, dominates the higher modes. Detailed results are presented for a single well, a pair of identical circular wells, and linear and equilateral triplets. Comparison with Molin's (2001) result for a rectangular well suggests that the present result for a circular well should provide a good approximation for the Helmholtz mode in any well of the same cross-sectional area and moderate aspect ratio.

1. Introduction

I consider here the slow oscillations ($\omega^2 \ll g/a$, where a is a characteristic radius) of a liquid in a set of N cylindrical wells of depth h_n and cross-section $S_n \equiv \pi a_n^2$ that are bounded above by free surfaces and below by a common, semi-infinite reservoir. The results may be of practical interest in connection with artesian wells or marine operations (cf. Molin 2001). The problem for a single well resembles that for a bottomless harbour (Garrett 1970). The problem for twin wells goes back to Newton (1686), who showed that the natural period of oscillations in a U-tube is equal to that of a simple pendulum with a length equal to half the length of the liquid in the U-tube.

Let a be a representative radius and h a representative depth. The assumption $\omega^2 \ll g/a$, which requires $h \gg a$, permits the motion in each well to be described by the dominant mode, for which the fluid moves as a rigid body. The neglect of the higher modes† prevents the local matching, in the mouth of the well, of the solution in the well to that in the reservoir, but closure may be effected, and the eigenvalues ($\kappa \equiv \omega^2/g$) approximated, by matching the total impulse (cf. Lamb 1932, § 196).

I formulate the N-well problem in § 2 and develop the solution for a single well in § 3. Comparison with Molin's (2001) result for a rectangular well suggests that the present result for a circular well should provide a good approximation for the Helmholtz mode in any well of the same cross-sectional area and moderate aspect ratio.

In § 4, I consider a pair of identical circular wells. There then are two normal modes. In the slower of these modes, the motion is symmetric with respect to the mid-plane, and the solution in the reservoir is source-like. In the faster mode, the motion is antisymmetric with respect to the mid-plane, and the solution in the

† The higher modes for a circular well correspond to the positive zeros of $J_m'(k_{mn}a)$, while the dominant mode corresponds to $m = 0$ and $k_{00} = 0$.

reservoir is dipole-like. The symmetric mode resembles that for a single well and may be described by the modifier *Helmholtz*. The antisymmetric mode resembles that for a U-tube and may be described by the modifier *sloshing*.

In § 5, I consider a linear array of N identical wells and display explicit results for $N = 3$. In § 6, I consider N identical wells at the vertices of an N-sided regular polygon and display explicit results for $N = 3$.

2. The boundary-value problem

Consider a set of N wells ($n = 1, 2, \dots, N$) of cross-section $S_n \equiv \pi a_n^2$ and depth h_n (mouths at $z = 0$, free surfaces at $z = h_n$). The solution of Laplace's equation for the velocity potential ϕ_n in the nth well, subject to the boundary conditions

$$\partial_z \phi_n = w_n \quad (z = 0), \quad \partial_z \phi_n = (\omega^2/g)\phi_n \quad (z = h_n), \tag{2.1a,b}$$

is given by

$$\phi_n = w_n(z - h_n + \kappa^{-1}) \quad (0 \leqslant z \leqslant h_n), \quad \kappa \equiv \omega^2/g, \tag{2.2a,b}$$

where w_n is the unknown velocity in the nth well and is implicitly simple harmonic with frequency ω. The corresponding solution in the reservoir, subject to (2.1a) and a null condition for $z \downarrow -\infty$, is given by

$$\phi(\boldsymbol{r}) = \sum_{n=1}^{N} \phi_n(\boldsymbol{r}) \quad (z < 0), \quad \phi_n(\boldsymbol{r}) = \frac{w_n}{2\pi} \iint_{S_n} \frac{\mathrm{d}S(\boldsymbol{\rho}_n)}{|\boldsymbol{r} - \boldsymbol{\rho}_n|}. \tag{2.3a,b}$$

Equating the total impulse, $\iint \phi \, \mathrm{d}S$ (density omitted), given by (2.2a) to that given by (2.3a) in the mth mouth, we obtain the N linear, homogeneous equations

$$\sum_{n=1}^{N} [\delta_{mn}(h_n - \kappa^{-1}) + A_{mn}] a_n w_n = 0 \quad (m = 1, \dots, N), \tag{2.4}$$

where

$$A_{mn} \equiv \frac{1}{2\pi(S_m S_n)^{1/2}} \iint \mathrm{d}S(\boldsymbol{r}_n) \iint \frac{\mathrm{d}S(\boldsymbol{\rho}_m)}{|\boldsymbol{r}_n - \boldsymbol{\rho}_m|} = A_{nm}, \tag{2.5}$$

δ_{mn} is the Kronecker delta, and $a_n \equiv (S_n/\pi)^{1/2}$.

For a circular well of radius a_n

$$A_{nn} \equiv A_n = 8a_n/3\pi \tag{2.6}$$

(cf. Rayleigh 1896, § 3.12; (2.6) is an upper bound for a non-circular cross-section (Pólya & Szegö 1951)).

If the inter-axial distance $b_{mn} \gg a_n$, (2.5) may, but need not, be approximated by

$$A_{mn} \simeq a_m a_n / 2b_{mn} \quad (m \neq n). \tag{2.7}$$

3. Single well

For a single circular well of radius a and depth h, (2.2a) reduces to

$$\phi = w(z - h + \kappa^{-1}) \quad (0 < r < a, \quad 0 < z < h), \tag{3.1}$$

(2.3b) exhibits the source-like asymptotic behaviour

$$\phi \sim \tfrac{1}{2}a^2 w R^{-1}, \quad R \equiv (x^2 + y^2 + z^2)^{1/2} \to \infty, \tag{3.2a,b}$$

and (2.4) implies

$$\kappa = \frac{\omega^2}{g} = \frac{1}{h + \alpha a}, \quad \alpha \equiv \frac{8}{3\pi} = 0.849. \qquad (3.3a, b)$$

This compares with Molin's result (28) for a rectangular pool with sides b and l, which we place in the form (3.3) with $a = (bl/\pi)^{1/2}$, the equivalent radius of the rectangle. Molin obtains

$$\sqrt{\pi}\alpha = \left(\frac{b}{l}\right)^{1/2} \sinh^{-1}\frac{l}{b} + \left(\frac{l}{b}\right)^{1/2} \sinh^{-1}\frac{b}{l} + \frac{1}{3}\left[\left(\frac{b}{l}\right)^{3/2} + \left(\frac{l}{b}\right)^{3/2} - \left(\frac{b}{l} + \frac{l}{b}\right)^{3/2}\right], \qquad (3.4)$$

which has a maximum of $0.839\sqrt{\pi}$ for $b/l = 1$, is symmetric under $b/l \to l/b$, and is within 4% of (3.3b) for $\frac{1}{2} < b/l < 2$ or 11% for $\frac{1}{4} < b/l < 4$. This suggests that (3.3) should be a good approximation for any well of cross-sectional area πa^2 and moderate aspect ratio.

4. Twin wells

Now suppose that $N = 2$ and the wells are identical circular cylinders of radius a, depth h, and axes at $x = \pm\frac{1}{2}b$ ($b > 2a$). Then

$$A_{11} = A_{22} \equiv A = \frac{8a}{3\pi}, \quad A_{12} = A_{21} \equiv B, \qquad (4.1a, b)$$

where B is given by (2.5) and admits the expansion

$$B = \frac{a^2}{2b}\left[1 + \frac{1}{4}\frac{a^2}{b^2} + O\left(\frac{a^4}{b^4}\right)\right]. \qquad (4.2)$$

Substituting (4.1a, b) and $N = 2$ into (2.4), we obtain

$$(h - \kappa^{-1} + A)w_1 + Bw_2 = 0, \quad Bw_1 + (h - \kappa^{-1} + A)w_2 = 0. \qquad (4.3a, b)$$

The Helmholtz and sloshing modes are given by

$$1/\kappa_1 = h + A + B, \quad w_1 = w_2 = w, \qquad (4.4a, b)$$

and

$$1/\kappa_2 = h + A - B, \quad w_1 = -w_2 = w. \qquad (4.5a, b)$$

The Helmholtz mode exhibits the source-like asymptotic behaviour (cf. (3.2))

$$\phi_1 + \phi_2 \sim \frac{wa^2}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \sim \frac{wa^2}{R} \quad (z < 0, \quad R \to \infty), \qquad (4.6)$$

where

$$R \equiv (x^2 + y^2 + z^2)^{1/2}. \qquad (4.7)$$

The sloshing mode exhibits the dipole-like asymptotic behaviour

$$\phi_1 + \phi_2 \sim \frac{a^2 w}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \sim \frac{a^2 bwx}{2R^3} \quad (z < 0, \quad R \to \infty). \qquad (4.8)$$

The equivalent-pendulum length for the sloshing mode, $h + A - B$, differs from that for Newton's U tube (half the total length of the fluid in the tubes) in the corrections for the open end and mutual coupling.

5. Linear array

Now suppose that the wells form a rectilinear array of equally spaced, identical circular cylinders. Then $A_{nn} = A$ is given by (2.6),

$$A_{mn} \equiv B_{|m-n|} \quad (m \neq n) \tag{5.1}$$

depends only on $|m-n|$, rather than separately on m and n, and (2.4) reduces to the Toeplitz system (see Golub & Van Loan 1996)

$$\sum_{n=1}^{N} [\delta_{mn}(h + A - \kappa^{-1}) + (1-\delta_{mn})B_{|m-n|}]w_n = 0. \tag{5.2}$$

Letting $N = 3$, we obtain the matrix equation

$$\begin{bmatrix} \theta & B_1 & B_2 \\ B_1 & \theta & B_1 \\ B_2 & B_1 & \theta \end{bmatrix} \{w_n\} = 0, \quad \theta \equiv h + A - \kappa^{-1}. \tag{5.3a,b}$$

The corresponding determinantal equation is

$$\Delta(\theta) = (\theta - B_2)(\theta^2 + B_2\theta - 2B_1^2) = 0. \tag{5.4}$$

The root $\theta = B_2$ yields the sloshing mode,

$$\kappa^{-1} = h + A - B_2, \quad [w_n] = [1, 0, -1]w_1, \tag{5.5a,b}$$

which, since the middle well is quiescent, is equivalent to (4.5). The remaining roots, for which $\theta = \theta_{\pm} = -\frac{1}{2}B_2 \pm (2B_1^2 + \frac{1}{4}B_2^2)$, yield the Helmholtz modes

$$\kappa^{-1} = h + A - \theta_{\pm}, \quad [w_n] = \frac{3}{2}\left(1 - \frac{2B_1}{\theta_{\pm}}\right)^{-1}\left[1, -\frac{2B_1}{\theta_{\pm}}, 1\right]\bar{w}, \tag{5.6a,b}$$

where $\bar{w}$ is the mean displacement. The approximation (2.7) yields

$$B_1 \simeq \frac{a^2}{2b}, \quad B_2 = \frac{a^2}{b}, \quad \theta_{\pm} = B_1(-1 \pm \sqrt{3}). \tag{5.7a–c}$$

6. Polygonal configurations

The Toeplitz system (5.2) also holds for a set of N circular wells of radius a and depth h at the vertices of an N-sided regular polygon of side b, but the inter-axial spacing is given by, and $B_{|m-n|}$ must be calculated for,

$$\frac{b\ \sin[|m-n|(\pi/N)]}{\sin(\pi/N)} \equiv b_{|m-n|}. \tag{6.1}$$

Letting $N = 3$, we obtain

$$b_{|m-n|} = b, \quad B_{|m-n|} = B, \tag{6.2a,b}$$

where $B = A_{12}$ is given by (2.5), and (5.2) reduces to

$$\begin{vmatrix} \theta & B & B \\ B & \theta & B \\ B & B & \theta \end{vmatrix} [w_n] = 0, \tag{6.3}$$

where θ is defined by (5.3b). The corresponding determinantal equation is

$$\Delta(\theta) = \theta^3 - 3B^2\theta + 2B^3 = (\theta - B)^2(\theta + 2B) = 0. \tag{6.4}$$

The Helmholtz mode corresponds to $\theta = -2B$, for which

$$\kappa^{-1} = h + A + 2B, \quad [w_n] = [1, 1, 1]\bar{w}. \tag{6.5a, b}$$

The sloshing modes correspond to the double root $\theta = B$, for which

$$\kappa^{-1} = h + A - B, \quad [w_n] = [1, 0, -1]w_1 + [0, 1, -1]w_2. \tag{6.6a, b}$$

We remark that the present triplet comprises one Helmholtz mode and two sloshing modes, in contrast to the converse for the rectilinear triplet.

This work was supported in part by the Division of Ocean Sciences of the National Science Foundation Grant OCE98-03204, and by the Office of Naval Research Grant N00014-92-J-1171.

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